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Stable D-branes, calibrations and generalized Calabi-Yau geometry

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Abstract: We introduce generalized calibrations that take into account the gauge field on the D-brane so that calibrated submanifolds minimize the Dirac-Born-Infeld energy. We establish the calibration bound and show that the calibration form is closed in a supersymmetric background with non-vanishing NS-NS 3-form H and dilaton Φ. We show that the calibration conditions are equivalent to the existence of unbroken supersymmetry on the D-brane. We study the problem of supersymmetric D-branes in the presence of $H \neq 0$ also from the world-sheet approach and find exactly the same conditions. Finally, we show that our notion of generalized calibrations is equivalent to the calibrations introduced in the context of generalized Calabi-Yau geometry in math.DG/0401221.

Keywords: D-branes, Superstring Vacua, Differential and Algebraic Geometry.

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Contents

JHEP08(2005)099

1. Introduction

String/M-theory on supersymmetric backgrounds with non-vanishing fluxes is currently a very active field of study. One reason is that those backgrounds provide the setup for models with attractive phenomenology and another is that they appear in generalizations of the AdS/CFT correspondence. The background geometry in this paper consists of non-vanishing fields in the common NS-NS sector of type IIA and IIB supergravities, i.e. we consider a non-vanishing dilaton Φ and 3-form H, but put all R-R fields and fermions to zero. The supersymmetry conditions for backgrounds with fluxes, pioneered in [1], lead to G-structures. We will mainly consider geometries with $\mathrm{SU}(n)_L \times \mathrm{SU}(n)_R$-structure, where the $\mathrm{SU}(n)_{L/R}$ are constant with respect to covariant derivatives with different connections $\nabla \pm \frac{1}{2}H$ [2].

In this paper we are interested in the conditions for branes to preserve some of the supersymmetry of the background. In the simplest case, without fluxes, the background has special holonomy and supersymmetric branes wrap calibrated submanifolds [3], which are volume-minimizing [4–6]. For $\mathrm{SU}(n)$ holonomy (Calabi-Yau) there are two cases depending on whether the calibration is $e^{i\omega}$ or $\Re(\Omega)$, where ω is the Kähler form and Ω is the $(n,0)$-form. These correspond to complex and special lagrangian submanifolds respectively.

In supersymmetric backgrounds with fluxes, supersymmetric branes are associated with generalized calibrations, which were introduced in [7] and extensively studied in [8, 9]. These calibrations take into account the coupling of branes with background fluxes so that the calibrated submanifolds are no longer volume-minimizing but rather energy-minimizing. Here we introduce another notion of generalized calibrations, in the same general philosophy though, which takes into account the gauge field $\mathcal{F}$ on a D-brane. As far as the author is aware a calibration like this has not yet been introduced for general Dp-branes (see [10] for a brief discussion of the case of the D4-brane as dimensional reduction of the M5-brane). Generalized calibrations now minimize the Dirac-Born-Infeld energy. Furthermore it is shown that the calibration conditions are equivalent to the vanishing of the gluino supersymmetry transformation for some spinors. The conditions for the latter were studied in [11].

However, we can study these conditions also from the string world-sheet viewpoint where D-branes are regarded as boundary conditions for open strings. In the case of vanishing 3-form flux and flat gauge field on the D-brane, $\mathcal{F} = 0$, it is well-known that this approach gives equivalent results [12, 5]. The string world-sheet approach starts from an $N = (2, 2)$ SCFT in the bulk, which induces $\mathrm{U}(n)_L \times \mathrm{U}(n)_R$ structure, and demands that the boundary conditions preserve $N = 2$ world-sheet supersymmetry. This is precisely the condition for the D-branes to descend to topological string theory so they are called topological branes. Depending on which combination of left- and right-moving supersymmetry is preserved one has B-type and A-type D-branes corresponding to the complex and special lagrangian submanifolds of the effective action approach respectively.

In [13] it was discovered that there exist supersymmetric D-branes of type A which are not special lagrangian if the gauge field $\mathcal{F}$ is turned on. In that paper the condition for the D-branes to be topological was worked out: they are coisotropic rather than lagrangian. However, the requirement of $N = 2$ world-sheet supersymmetry alone is not enough for target space supersymmetric D-branes. To proceed one should note that target space supersymmetry is generated by the spectral flow operators. In order to globally define these spectral flow operators and thus have preserved target space supersymmetry in the bulk we must further reduce the structure to $\mathrm{SU}(n)_L \times \mathrm{SU}(n)_R$. On the boundary, one needs preservation of the spectral flow operator, which is called the stability condition. In the simplest case of $\mathcal{F} = 0$ stability corresponds to the requirement of *special* lagrangian in addition to just lagrangian. In [14] this stability condition was studied in the case of non-vanishing gauge field on a D-brane in a Calabi-Yau manifold ($H = 0$) and shown to be completely equivalent to the conditions for supersymmetric D-branes found from the effective action approach in [11].

In this paper we generalize the world-sheet approach to the case $H \neq 0$. The topological string theory with $H \neq 0$ was introduced in [15] and the condition for the D-brane to be topological was studied in [16]. Here we construct the remaining condition for the D-brane to be stable and show that both requirements, topological and stable, are exactly the same as the conditions for the D-brane to be generalized calibrated. Therefore, also in the case $H \neq 0$ we find the same supersymmetry requirements from the world-sheet approach as from the effective action approach.

JHEP08(2005)099

A geometry with $\mathrm{U}(n)_L \times \mathrm{U}(n)_R$-structure where the $\mathrm{U}(n)$ structures are covariantly constant with respect to different connections $\nabla \pm \frac{1}{2}H$, called bihermitian geometry [2], is in fact completely equivalent to a (twisted) generalized Kähler structure, introduced in [17], building on the work of [18]. A generalized Kähler structure consists of two commuting (twisted) generalized complex structures $(\mathcal{J}_1, \mathcal{J}_2)$. If the structure is further reduced to $\mathrm{SU}(n)_L \times \mathrm{SU}(n)_R$ we call it a generalized Calabi-Yau geometry[1]. It comes as no surprise then that our generalized calibrations should have an interpretation in the theory of generalized complex structures. In [16] it was shown that a D-brane is topological if it is a generalized complex submanifold with respect to $\mathcal{J}_1$ (for type B) or $\mathcal{J}_2$ (for type A). Furthermore, in [17] a definition of a calibration in a generalized Calabi-Yau geometry was given. We show that our concept of generalized calibration is equivalent i.e. a brane is generalized calibrated (in the sense this paper) if it is a generalized complex submanifold with respect to $\mathcal{J}_1$ and calibrated (in the sense of [17]) with respect to $\mathcal{J}_2$ for type B and vice-versa for type A. Under the mirror symmetry automorphism of the world-sheet theory $\mathcal{J}_1$ and $\mathcal{J}_2$ are exchanged so that mirror symmetry indeed swaps B- and A-type branes. Furthermore, we note that B-type topological string theory defined in [15] only sees $\mathcal{J}_1$ while the stability condition of the B-brane depends on $\mathcal{J}_2$ and vice-versa for the A-brane. This is in fact also the case for $H = 0$, where the roles of complex structure and Kähler form are exchanged between the topological field theory dependence and the stability criteria.

Other work on generalized complex structures from the target space viewpoint is [19 – 21], from the world-sheet viewpoint [22 – 24] and on the relation with mirror symmetry [25 – 28].

In section 2 a definition of generalized calibrations is given. The calibrated submanifolds minimize the Dirac-Born-Infeld energy. A suitable calibration form is constructed from the generators of unbroken supersymmetry. It is shown that this form is closed and generates the calibration bound. We show that the conditions for saturating the bound coincide with the condition for supersymmetric cycles. In section 3 the same conditions for supersymmetry are found, but now from the world-sheet approach. In section 4 we present the canonical example of ordinary Calabi-Yau manifolds. In section 5 the results are interpreted in the context of generalized Calabi-Yau geometry.

2. Calibrations

2.1 Generalized calibrations

In this subsection we will quickly review the concept of calibrations and generalize it slightly to include the gauge field living on the world-volume of D-branes. Calibrations were introduced in [3] in order to construct volume minimizing submanifolds.

An oriented tangent p-plane is a vector subspace V of T_xM with an orientation. A p-form ϕ on a riemannian manifold (M, g) is a *calibration* if $d\phi = 0$ (ϕ is closed) and for

[1] Note that the definition of a generalized Calabi-Yau structure in [18] is different from that in [17]. Here we mean the latter stronger one.

JHEP08(2005)099

any tangent p-plane, V, it satisfies

$$\phi|_V \leq \mathrm{vol}|_V \,, \tag{2.1}$$

where $\phi|_V$ is the pull-back to V and $\mathrm{vol}|_V$ is the induced volume form on V. We also demand that in every point x of M, there exist p-planes for which the bound is saturated. Those p-planes form the *contact set*.

A p-dimensional submanifold N of M — a p-brane — is calibrated by ϕ if at any point $x \in N$ it satisfies $\phi|_{T_x N} = \mathrm{vol}|_{T_x N}$ i.e. it saturates the bound at any point x. In this paper we will often rewrite this condition as

$$P(\phi)_\epsilon = \sqrt{P(g)} \,, \tag{2.2}$$

where P denotes the pullback to the p-brane world-volume and[2]

$$P(\phi)_\epsilon = \frac{1}{p!} P(\phi)_{a_1 \ldots a_p} \epsilon^{a_1 \ldots a_p} \,. \tag{2.3}$$

It is clear that such branes are world-volume minimizing within their homology class since if we take another brane N' within the same class $N' = N + \partial Q$ we find

$$\mathrm{vol}(N') = \int_{N'} \mathrm{vol} \geq \int_{N'} \phi = \int_N \phi + \int_Q d\phi = \int_N \phi = \int_N \mathrm{vol} = \mathrm{vol}(N) \,, \tag{2.4}$$

where we used Stokes' theorem and $d\phi = 0$. Calibrations are often constructed from bilinears in spinors [29]. One can then make the link with supersymmetry generators and show that calibrated branes preserve some of the supersymmetry of the background.

In [7] generalized calibrated submanifolds were introduced which do not minimize the volume but rather the brane energy, which includes couplings to the background fields. Likewise these branes wrap supersymmetric cycles. In this paper, however, we specialize to D-branes and also take into account the gauge field $\mathcal{F}$, with $d\mathcal{F} = 0$, on the D-brane. The basic philosophy of generalized calibrations is the same in that we will now minimize the D-brane energy i.e. the Dirac-Born-Infeld energy. A D-brane is now a generalized submanifold with data $(N, \mathcal{F})$ with $\mathcal{F}$ an abelian gauge field. We introduce a sum of forms of different dimension $\phi \in \wedge^\bullet T^*_M$,

$$\phi = \sum_l \phi^{(l)} \,, \tag{2.5}$$

and a polynomial in $\mathcal{F}$, $\mathrm{pol}(\mathcal{F})$, in which the products are wedges. ϕ is a *generalized calibration* if $d\phi = 0$ and for every generalized submanifold $(N, \mathcal{F})$ the following bound is satisfied:

$$\left(P(\phi) \wedge \mathrm{pol}(\mathcal{F})\right)_{[p],\epsilon} \leq e^{-\Phi} \sqrt{P(g-b) + \mathcal{F}} \,, \tag{2.6}$$

where we selected out the p-dimensional part of $P(\phi) \wedge \mathrm{pol}(\mathcal{F})$, b is the NS-NS field and Φ the dilaton. It will be convenient to introduce $F = \mathcal{F} - P(b)$, since $\mathcal{F}$ and $P(b)$ will always appear in this combination in D-brane actions. As usual the torsion is given by

[2] See appendix A for more conventions.

JHEP08(2005)099

$H = db$. The right-hand side of the bound is indeed the Dirac-Born-Infeld energy. We had to go to the trouble of considering ϕ separately from $\mathcal{F}$ because the form ϕ is defined on M while $\mathcal{F}$ is only defined on the D-brane world-volume and in fact part of the D-brane data. The reader should keep in mind that this concept of generalized calibrations is different from [7]. We will use the term generalized calibrations in the hope that it will cause no confusion. Now $(N, \mathcal{F})$ is a *generalized calibrated submanifold* if

$$(P(\phi) \wedge \mathrm{pol}(\mathcal{F}))_\epsilon = e^{-\Phi}\sqrt{P(g) + F}\,. \tag{2.7}$$

If we now take another D-brane $(N', \mathcal{F}')$ where $N' = N + \partial Q$ is in the same homology class as N we can go through the same reasoning as in (2.4) to show that $(N, \mathcal{F})$ has indeed minimal energy within its class provided that

$$\int P_{N'}(\phi) \wedge \mathrm{pol}(\mathcal{F}') = \int P_N(\phi) \wedge \mathrm{pol}(\mathcal{F})\,. \tag{2.8}$$

The exact topological condition for this statement to be true is that there must exist a gauge bundle on Q such that its Chern class restricted to N gives the Chern class of $\mathcal{F}$ and its Chern class restricted N' the Chern class of $\mathcal{F}'$. It might be better to choose a gauge bundle on the whole manifold M right from the start (the choice of a particular gauge field $\mathcal{F}$ within that bundle is still free as it should be since it is part of the data of the D-brane) although we loose some generality then[3].

2.2 The calibration form

In this subsection we construct the calibration form ϕ and show that it is closed while in the next subsection we will prove the bound (2.6). The basic ingredients of our calibration form are the generators of left- and right-moving preserved supersymmetry transformations. The supersymmetry transformations for type-II theories contain two 10-dimensional Majorana-Weyl spinor parameters ϵ_L and ϵ_R. Here, L and R indicate whether they originate from left- or right-moving supersymmetry generators on the string world-sheet. In type-IIA these spinors have opposite chirality while in type-IIB they have the same chirality. In type-II supergravity the supersymmetry transformations for the gravitino and dilatino read respectively:

$$\begin{aligned}
\delta\psi_{L/R\,\mu} &= \left(\nabla_\mu \pm \frac{1}{4}\not{H}_\mu\right)\epsilon_{L/R} = \nabla_{\pm\,\mu}\epsilon_{L/R}\,,\\
\delta\lambda_{L/R} &= \left(\not{\partial}\Phi \pm \frac{1}{2}\not{H}\right)\epsilon_{L/R}\,,
\end{aligned} \tag{2.9}$$

where L gets the plus sign and R the minus sign, ∇ is the covariant derivative containing the Levi-Civita connection, Φ is the dilaton, H the NS-NS 3-form and all R-R forms were put to zero. We consider geometries with both left- and right-moving preserved supersymmetries generated by ϵ_L and ϵ_R respectively. The vanishing of the gluino supersymmetry transformation on the brane will then relate ϵ_L and ϵ_R.

[3] We thank Jim Bryan for explaining all this.

JHEP08(2005)099

If we introduce the sum of forms $\phi_0 \in \wedge^\bullet T^*_M$,

$$\phi_0 = \sum_{l=0}^{10} \frac{1}{l!} \bar{\epsilon}_R \gamma_{\mu_1 \dots \mu_l} \epsilon_L \, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_l} \,, \tag{2.10}$$

we find using both $\delta\psi_{L/R\,\mu} = 0$ and $\delta\lambda_{L/R} = 0$ for ϵ_L and ϵ_R (see also [30, 20])

$$d\phi_0 - H \wedge \phi_0 - d\Phi \wedge \phi_0 = 0 \,. \tag{2.11}$$

Therefore we should take our candidate generalized calibration to be

$$\phi = e^{-\Phi} \phi_0 e^{-b} \,, \tag{2.12}$$

so that it is closed. Furthermore we take $\mathrm{pol}(\mathcal{F}) = e^{\mathcal{F}}$ so that in the pull-back to the world-volume $P(b)$ and $\mathcal{F}$ indeed appear in the invariant combination $F = \mathcal{F} - P(b)$.

To proceed we like to consider supersymmetric cycles in euclidean geometry so we split our space-time manifold as $\mathbb{R}^{1,9-d} \times M$ with Minkowski metric on $\mathbb{R}^{1,9-d}$, euclidean metric g on the d-dimensional internal manifold, H only non-vanishing on M and everything independent of the coordinates in $\mathbb{R}^{1,9-d}$. We can then restrict ourselves to studying the euclidean geometry of the internal manifold M. The 10-dimensional Majorana-Weyl spinors $\epsilon_{L/R}$ decompose into spinors of $\mathbb{R}^{1,9-d}$ and spinors in the internal manifold. For instance in the case $d = 2n$ we find

$$\epsilon_{L/R} = \zeta_{L/R} \otimes \eta_{L/R} + \zeta^c_{L/R} \otimes \eta^c_{L/R} \tag{2.13}$$

for any $(1, 9-2n)$-dimensional Weyl-spinors $\zeta_{L/R}$, with $\zeta^c_{L/R}$ their Majorana conjugates. We also have $\eta^c_{L/R} = C(\eta_{L/R})^*$ such that the $\epsilon_{L/R}$ are Majorana in 10 dimensions. Note that when n is odd η and η^c have different chirality while when n is even they have the same chirality. Plugging (2.13) into (2.9) we find supersymmetry variations of exactly the same form but now for the $\eta_{L/R}$ and $\eta^c_{L/R}$. If $\eta_{L/R}$ and $\eta^c_{L/R}$ generate independently preserved supersymmetries[4] we can define

$$\phi_{0,i_1 \dots i_l} = \sum_{l=0}^{d} {\eta_R}^\dagger \gamma_{i_1 \dots i_l} \eta_L \,, \qquad \text{or} \qquad \phi_{0,i_1 \dots i_l} = \sum_{l=0}^{d} {\eta^c_R}^\dagger \gamma_{i_1 \dots i_l} \eta_L \tag{2.14}$$

and find that both also obey (2.11).

The case just presented, in which we have two preserved supersymmetries on the internal manifold on the left-moving side, generated by η_L and η^c_L and two preserved supersymmetries on the right-moving side generated by η_R and η^c_R, will be the most studied in this paper. Normalizing the spinors such that $\eta^\dagger_{L/R} \eta_{L/R} = 1$, ${\eta^c_{L/R}}^\dagger \eta^c_{L/R} = 1$ we can define

$$\omega_{L,ij} = -i \eta^\dagger_L \gamma_{ij} \eta_L \,, \qquad \omega_{R,ij} = -i \eta^\dagger_R \gamma_{ij} \eta_R \,, \tag{2.15a}$$

$$\Omega_{L,i_1 \dots i_n} = \eta^\dagger_L \gamma_{i_1 \dots i_n} \eta^c_L \,, \qquad \Omega_{R,i_1 \dots i_n} = \eta^\dagger_R \gamma_{i_1 \dots i_n} \eta^c_R \,. \tag{2.15b}$$

From this we can construct two almost complex structures $J_{L/R} = g^{-1} \omega_{L/R}$, $J^2_{L/R} = -\mathbf{1}$.

[4] This means there is no relation needed between η_L and η^c_L nor between η_R and η^c_R as would be the case in e.g. $\mathrm{Spin}(7)_L \times \mathrm{Spin}(7)_R$-structure.

JHEP08(2005)099

It is possible to show from the dilatino equation in (2.9) that the Nijenhuistensors vanish [1] so that $J_{L/R}$ are integrable. Note that $\eta_{L/R}$ and $\eta^c_{L/R}$ are the empty and completely filled state of eqs. (A.7) and (A.8) for $J_{L/R}$ respectively.

From the vanishing of the gravitino transformations we find furthermore

$$\nabla^{\pm}_i \omega_{jk} = \nabla^{\pm}_i \omega_{jk} \mp H_i{}^l{}_{[j}\omega_{|l|k]} = 0\,, \tag{2.16a}$$

$$\nabla^{\pm}_i \Omega_{j_1\ldots j_n} = \nabla^{\pm}_i \Omega_{j_1\ldots j_n} \mp \frac{n}{2} H_i{}^l{}_{[j_1}\Omega_{|l|j_2\ldots j_n]} = 0\,, \tag{2.16b}$$

i.e. the left- and right-moving tensors are covariantly constant with respect to the Bismut connections $\nabla^+ = \nabla + \frac{1}{2}H$ and $\nabla^- = \nabla - \frac{1}{2}H$ respectively. From the integrability of the complex structures $J_{L/R}$, their compatibility with the metric $gJ + J^T g = 0$ and (2.16a) follows that we have in fact bihermitian geometry (g, J_L, J_R, H) [2]. We will use its connection to the generalized Kähler structure of [18, 17] later in the paper. We have $\mathrm{U}(3)_L \times \mathrm{U}(3)_R$ structure which is further reduced to $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ structure by the existence of $\Omega_{L/R}$ satisfying (2.16b). Since eqs. (2.16a) and (2.16b) contain the Bismut connection instead of the Levi-Civita connection it does not follow that we have special holonomy. Only when $H = 0$ there is $\mathrm{SU}(n)$ holonomy and M is a Calabi-Yau manifold.

2.3 The bound and the supersymmetry variation of the gluino

In this subsection we establish the bound (2.6) for our candidate generalized calibration ϕ and show that the bound is saturated if and only if the gluino variation vanishes. In that case we say that the D-brane $(N, \mathcal{F})$ wraps a supersymmetric cycle.

Let us define the following γ-matrix structures

$$\begin{aligned}\rho(F) &= \sum_l \frac{1}{2^l l!(p-2l)!} F_{a_1a_2}\ldots F_{a_{2l-1}a_{2l}}\gamma_{a_{2l+1}\ldots a_p}\epsilon^{a_1\ldots a_p}\,,\\ \Gamma(F) &= \frac{1}{\sqrt{\det(P(g)+F)}}\rho(F)\,,\end{aligned} \tag{2.17}$$

with as before $F = \mathcal{F} - P(b)$. Using the methods of [31 – 33] we can show that

$$\rho(F)^\dagger \rho(F) = (\rho_E(F) + \rho_O(F))(\rho_E(F) - \rho_O(F)) = \det(P(g) + F)\,, \tag{2.18}$$

where $(\rho_E(F))^\dagger = \rho_E(F)$ and $\rho_O(F)^\dagger = -\rho_O(F)$ the hermitian and anti-hermitian part of $\rho(F)$. Alternatively we have $\Gamma(F)^\dagger\Gamma(F) = 1$.

These matrices are closely related to the Γ-matrix defined in [31 – 33]. That matrix plays a crucial role in the definition of the κ-symmetry and supersymmetry transformations for Dp-branes. In fact, for the Dp-branes we consider in this paper, i.e. the ones extended solely in the internal manifold with only magnetic fields turned on, we have

$$\Gamma_{\mathrm{IIA}} = -(\gamma_{11}\gamma_0\Gamma_E + \gamma_0\Gamma_O)\,, \qquad \Gamma_{\mathrm{IIB}} = -(\gamma_0\Gamma_E + \tau_3\gamma_0\Gamma_O)\tau_1\,. \tag{2.19}$$

These γ-matrix structures convert left-moving spinors into right-moving spinors. Indeed, in the IIA case Γ_{IIA} contains an odd number of γ-matrices so that it changes the chirality while in the IIB case τ_1 takes care of the switch. They also satisfy $\Gamma^\dagger = \Gamma$ and $\Gamma^2 = 1$.

JHEP08(2005)099

The gluino supersymmetry transformation in a certain κ-gauge consists of a supersymmetry transformation and a compensating κ-transformation. In [34] it is shown that the preserved supersymmetries must satisfy

$$(1-\Gamma)\epsilon = 0\,, \tag{2.20}$$

with $\epsilon = (\epsilon_L, \epsilon_R)$. For the part of the spinors on the internal manifold this translates into

$$\Gamma(F)\eta_L = e^{i\gamma}\eta_R\,, \tag{2.21}$$

with $e^{i\gamma}$ a constant phase.

Using eq. (2.18) we can, following [35], link the gluino supersymmetry condition (2.21) to the bound (2.6). Indeed

$$\begin{aligned}\det(P(g)+F) = \eta_L^\dagger \det(P(g)+F)\eta_L &= \sum_{\eta'} \left(\eta_L^\dagger \rho^\dagger(F)\eta'\right)\left(\eta'^\dagger \rho(F)\eta_L\right) = \sum_{\eta'} \left|\eta'^\dagger \rho(F)\eta_L\right|^2 \\ &\geq \left|\eta_R^\dagger \rho(F)\eta_L\right|^2 \geq \left(\Re\left(e^{-i\gamma}\eta_R^\dagger \rho(F)\eta_L\right)\right)^2,\end{aligned} \tag{2.22}$$

with γ the same constant as before. In the second line we have introduced an orthonormal complete set of spinors $\sum_{\eta'} \eta'\,\eta'^\dagger = 1$. In the end we find the bound

$$e^{-\Phi}\sqrt{\det(P(g)+F)} \geq \Re\left(e^{-i\gamma}e^{-\Phi}P(\phi_0)e^F\right)_\epsilon\,, \tag{2.23}$$

where ϕ_0 given by (2.14). Moreover, from (2.11) we know that $d\left(\Re\left(e^{-i\gamma}e^{-\Phi}\phi_0 e^{-b}\right)\right) = 0$ so that we have indeed constructed a generalized calibration.

The bound is saturated if and only if

$$\eta'^\dagger \rho(F)\eta_L = 0, \quad \forall \eta' \neq \eta_R\,, \tag{2.24a}$$

$$\sqrt{\det(P(g)+F)} = e^{-i\gamma}\eta_R^\dagger \rho(F)\eta_L\,. \tag{2.24b}$$

These two conditions are completely equivalent to (2.21). It follows that this type of generalized calibrated D-branes is supersymmetric and vice-versa every supersymmetric D-brane is a generalized calibrated D-brane of this type.

Another related viewpoint on supersymmetry vs. calibrations made of bilinears of spinors is based on central charges in the supersymmetry algebra [36−39]. The calibration bound is then the well-known BPS bound and when there is unbroken supersymmetry the hamiltonian is equal to the central charge. This approach is heavily used in [9]. We defer working out the details for the calibrations at hand to further work.

3. World-sheet approach

In this section we consider the conditions for unbroken target space supersymmetry again, but now in the string world-sheet approach. We will find exactly the same conditions (2.24a) and (2.24b). The special case of $H = 0$ was already studied from this viewpoint in [14].

JHEP08(2005)099

Here we study an $N = (1, 1)$ non-linear sigma-model with bulk metric g and torsion $H = db$. As has been found in [12] and later studied in great detail in [40, 41], if we introduce a D-brane $(N, \mathcal{F})$, the gluing conditions read

$$\psi_R^i = R^i{}_j \psi_L^j \,, \tag{3.1}$$

with

$$R = P_+ \frac{g - F}{g + F} P_+ - P_- \,. \tag{3.2}$$

Here the submanifold N on which the D-brane wraps is determined by the integrable product structure $r = P_+ - P_-$, $r^2 = \mathbf{1}$, which is compatible with the metric, i.e. $r^T g r = g$. P_+, satisfying $P_+^2 = P_+$, projects on vectors tangential to the D-brane, while P_-, satisfying $P_-^2 = P_-$, projects on vectors normal to the D-brane. We also have

$$F = \mathcal{F}_M - P_+ b P_+ \,, \tag{3.3}$$

where $\mathcal{F}_M$ is a smooth extension of $\mathcal{F}$ to M, i.e. $P(\mathcal{F}_M) = \mathcal{F}$, satisfying $P_- \mathcal{F}_M = \mathcal{F}_M P_- = 0$.

We can promote the $N = (1, 1)$ supersymmetry to an $N = (2, 2)$ supersymmetry if and only if the target space manifold M admits a bihermitian geometry (g, J_L, J_R, H) [2]. The U(1) R-currents of the $N = (2, 2)$ geometry read

$$j_L = \omega_{L,ij} \psi_L^i \psi_L^j \,, \qquad j_R = \omega_{R,ij} \psi_R^i \psi_R^j \,. \tag{3.4}$$

If the D-brane is to preserve $N = 2$ supersymmetry we must have $j_L = j_R$ on the boundary. Using eq. (3.1) we find that we must have

$$R^T \omega_R R = \omega_L \,, \tag{3.5}$$

or alternatively

$$R^{-1} J_R R = J_L \,. \tag{3.6}$$

Plugging (2.15a) into (3.5) the condition becomes

$$\eta_R^\dagger R^i{}_k R^j{}_l \gamma_{ij} \eta_R = \eta_L^\dagger \gamma_{kl} \eta_L \,. \tag{3.7}$$

Before proceeding, we will first show that the $\Gamma(F)$ defined in (2.17) is in fact the spinor representation of R. We rewrite $\Gamma(F)$ as:

$$\Gamma(F) = \frac{\sqrt{\det(P(g))}}{\sqrt{\det(P(g) + F)}} \, \mathrm{se}(-\not{F}) \Gamma_N \,, \tag{3.8}$$

with "se" the skew-exponential function (the usual exponential function but with γ-matrices completely symmetrized at every order):

$$\mathrm{se}(\not{F}) = \sum_{l=0}^{\lfloor p/2 \rfloor} \frac{1}{2^l l!} \gamma^{a_1 \ldots a_{2l}} F_{a_1 a_2} \ldots F_{a_{2l-1} a_{2l}} = \not{e}^F \,, \tag{3.9}$$

JHEP08(2005)099

and

$$\Gamma_N = \frac{1}{p!\sqrt{\det(P(g))}}\epsilon^{a_1\dots a_p}\gamma_{a_1\dots a_p} \,. \tag{3.10}$$

In [34] it is shown that

$$\frac{\sqrt{\det(P(g))}}{\sqrt{\det(P(g)+F)}}\mathrm{se}\,(-\not{F}) = \exp\left[-\frac{1}{4}\phi^{ab}\gamma_{ab}\right], \tag{3.11}$$

with $\phi = 2\arctan F = \ln\frac{P(g)+F}{P(g)-F}$. This is a rotation with angle matrix ϕ in the spinor representation. On the other hand, $\Gamma_N(\gamma_{d+1})^{p+1}$ is the spinor representation of a reflection in the directions normal to the D-brane. Taking both together we find that if we define the spinor representation U_R as

$$R^i{}_j\gamma_i = U_R^\dagger\gamma_j U_R \,, \tag{3.12}$$

then

$$U_R^\dagger = U_R^{-1} = \Gamma(F)\,(\gamma_{d+1})^{p+1} \,. \tag{3.13}$$

Picking up where we left off at eq. (3.7) and plugging in (3.12) we find

$$(U_R\eta_R)^\dagger\,\gamma_{kl}\,(U_R\eta_R) = \eta_L^\dagger\gamma_{kl}\eta_L \tag{3.14}$$

Using eq. (3.13) and rephrasing (2.24a) as

$$\Gamma(F)\eta_L = e^{i\beta(\sigma)}\eta_R, \tag{3.15}$$

we see that condition (2.24a) implies (3.14) and thus (3.5). Since both η_L and η_R are normalized, the proportionality factor $e^{i\beta(\sigma)}$ can indeed only be a phase. At this point, it may still vary over the D-brane though. We emphasize this by indicating that it can be a function of the D-brane world-volume coordinates σ. It is condition (2.24b) that will fix it to a constant phase $\beta(\sigma) = \gamma$.

Conversely, from (3.6) follows that for every vector v that is a $(+i)$-eigenvalue of J_L, $w = Rv$ is a $(+i)$-eigenvalue of J_R, and vice-versa. Now η_R is the spinor that is annihilated by all $w^i\gamma_i$ with w a $(+i)$-eigenvalue of J_R. Using eq. (3.12) it follows that $U_R\eta_R$ is annihilated by all $v^i\gamma_i$ with v a $(+i)$-eigenvalues of J_L. This implies $U_R\eta_R \propto \eta_L$ or equivalently (3.15).

Summarizing we find:

$$\begin{aligned} R^{-1}J_R R = J_L &\Leftrightarrow \eta'^\dagger\rho(F)\eta_L = 0\,, \qquad \forall\eta' \neq \eta_R\,,\\ R^{-1}J_R R = -J_L &\Leftrightarrow \eta'^\dagger\rho(F)\eta_L = 0\,, \qquad \forall\eta' \neq \eta_R^c\,, \end{aligned} \tag{3.16}$$

The second statement can be proven analogously or just by noting that changing $J \to -J$ will indeed send $\eta \to \eta^c$. For later use we also note that

$$\rho(F)\eta_L \propto \eta_R \Leftrightarrow R^{-1}J_R R = J_L \Leftrightarrow R^{-1}(-J_R)R = -J_L \Leftrightarrow \rho(F)\eta_L^c \propto \eta_R^c\,. \tag{3.17}$$

From the point of view of topological string theory a boundary condition that preserves $N = 2$ supersymmetry is a *topological D-brane.* This is however not enough to have unbroken supersymmetry for the D-branes in target space. So, also from the world-sheet analysis

JHEP08(2005)099

we find a second condition, which we will derive now. The target space supersymmetry is generated by the spectral flow operators

$$S_{L/R} = \frac{1}{n!}\Omega_{L/R,i_1\ldots i_n}\psi^{i_1}_{L/R}\cdots\psi^{i_n}_{L/R}\,. \tag{3.18}$$

On the boundary we require matching of the spectral flow operators

$$S_R = e^{i\alpha}S_L\,. \tag{3.19}$$

We would like to show that this matching condition is equivalent to (2.24b) and find the precise relation between the phases $e^{i\alpha}$ and $e^{i\gamma}$. In order to do that, we introduce a charge conjugation matrix C (see (A.11) for the defining property) such that

$$\eta^c_L = C\eta^*_L\,. \tag{3.20}$$

Plugging in (2.15b) and using the same trick as above we rewrite the matching condition (3.19) as

$$(U_R\eta_R)^\dagger\,\gamma_{i_1\ldots i_n}U_R\eta^c_R = e^{i\alpha}\eta^\dagger_L\gamma_{i_1\ldots i_n}\eta^c_L \tag{3.21}$$

Suppose the condition (3.15) for having a topological brane is already satisfied. If we take the chirality of η_L to be positive we can rewrite it as

$$U_R\eta_R = e^{-i\beta(\sigma)}\eta_L\,. \tag{3.22}$$

Let us now calculate

$$\Gamma(F)\eta^c_L = \Gamma(F)C\eta^*_L = C\Gamma(F)^*\eta^*_L = e^{-i\beta(\sigma)}C\eta^*_R\,, \tag{3.23}$$

where we used (3.20), (A.11) and (3.15). We already know from (3.17) that this result should be proportional to η^c_R. However, there is some phase arbitrariness in the definition of η_R and thus also in the definition of γ in (2.24b). In order to find a definite relation between α and γ we fix it by choosing

$$\eta^c_R = C\eta^*_R\,, \tag{3.24}$$

with the *same* charge conjugation matrix as used for the left-movers, and find

$$\Gamma(F)\eta^c_L = e^{-i\beta(\sigma)}\eta^c_R\,. \tag{3.25}$$

From this follows

$$U_R\eta^c_R = (-1)^n e^{i\beta(\sigma)}\eta^c_L\,. \tag{3.26}$$

Plugging this into (3.21) we find that $\beta(\sigma) = \gamma$ should be constant and find the relation

$$e^{i\alpha} = (-1)^n e^{2i\gamma}\,. \tag{3.27}$$

Concluding, the world-sheet supersymmetry conditions (3.6) and (3.19) are exactly equivalent to the generalized calibration conditions (2.24a) and (2.24b) which are in turn equivalent to the target space gluino supersymmetry condition (2.21). The condition (3.19) is called the *stability* condition.

JHEP08(2005)099

4. Special case: Calabi-Yau manifold

In this section we specialize to the case $H = 0$, but non-vanishing F, and present the two canonical examples studied before in [11] from the target space perspective and in [14] from the world-sheet perspective. In this case we find:

$$\nabla J_{L/R} = d\omega = \nabla\Omega_{L/R} = d\Omega_{L/R} = \nabla\eta_{L/R} = 0\,, \tag{4.1}$$

where ∇ is the covariant derivative containing the Levi-Civita connection. So we have SU(3)-holonomy and M is a Calabi-Yau manifold. If we do not introduce any extra special holonomy we find that either $J_R = J_L$ or $J_R = -J_L$. Since mirror symmetry reverses the sign of J_R it will exchange these two cases.

4.1 B-branes: the complex case

In the first case of $J_R = J_L$, we find

$$R^{-1}JR = J, \tag{4.2}$$

from which follows $rJr = J$, which implies N is a complex submanifold with complex structure P_+JP_+, and F is of type $(1,1)$ on N. We must have $p = 2k$ even. We take $\eta_L = \eta_R$ and find for the calibration condition (2.24b):

$$\sqrt{\det(P(g)+F)} = e^{-i\gamma}\frac{1}{k!}(iP(\omega)+F)^k|_\epsilon = i^{k-n}e^{-i\alpha/2}\frac{1}{k!}(P(\omega)-iF)^k|_\epsilon\,. \tag{4.3}$$

For small F this reduces to the Donaldson-Uhlenbeck-Yau condition $g^{\alpha\bar\beta}F_{\alpha\bar\beta} = 0$. The case of $p = d$ was extensively studied in [35] (see also references therein). In there, the existence of a calibration bound like this was used as a constraint to calculate derivative and non-abelian corrections to the D-brane effective action.

4.2 A-branes: the coisotropic case

In the second case, we find

$$R^{-1}JR = -J\,. \tag{4.4}$$

As was first shown in [13] this leads to the following three properties (see also section 7.2 of [17]):

1. Let us define $\mathrm{Ann}\,T_N = \{\xi \in T^*_M : X^i\xi_i = 0, \forall X \in T_N\}$. Then we have $\omega^{-1}(\xi) \in T_N, \forall\xi \in \mathrm{Ann}\,T_N$. This means the submanifold is *coisotropic*. It also implies that the symplectic orthogonal bundle $T_N^\perp = \{Y \in T_M : \omega_{ij}X^iY^j = 0, \forall X \in T_N\}$ lies within T_N: $T_N^\perp \subseteq T_N$. Because ω is non-degenerate, the dimension of $T_N^\perp$ is the codimension of T_N.

2. $FY = 0$, $\forall Y \in T_N^\perp$ i.e. F descends to $T_N/T_N^\perp$.

3. $(\omega|_N)^{-1}F$ is an almost complex structure on $T_N/T_N^\perp$. In fact, in [13] it was shown that this complex structure is integrable.

JHEP08(2005)099

It can be shown that the complex dimension of $T_N/T_N^{\perp}$ should be even. This implies that $p - n = 2k$ is even. We take now

$$\eta_R = (-1)^{\frac{n(n-1)}{2}} \eta_L^c . \tag{4.5}$$

The phase factor comes about because we want to stick to our convention (3.24). In calculating it we used (A.13). For the calibration condition (2.24b) we find

$$\sqrt{\det(P(g) + F)} = e^{i\gamma} \left(P(\Omega) \wedge e^F \right)_{[p],\epsilon} . \tag{4.6}$$

5. Generalized Calabi-Yau geometry

In [17] it was shown that bihermitian geometry is equivalent to generalized Kähler structure. In the latter context, yet another notion of generalized calibrations was introduced. We will show however that these calibrations exactly coincide with the generalized calibrations studied here. First we start with a lightning review of generalized complex geometry and connect these concepts to the ideas of the previous sections as they are introduced. The reader is advised however to also consult [18] and [17].

5.1 Generalized complex geometry

In generalized (complex) geometry the usual statements about integrable subbundles of the tangent bundle T_M are replaced by similar statements about subbundles of $T_M \oplus T_M^*$. On this space there exists a natural metric defined by $(U, V) = (X+\xi, Y+\eta) = \frac{1}{2}\left(i_X\eta + i_Y\xi\right) = \frac{1}{2}\left(X^i\eta_i + Y^i\xi_i\right)$ with $X, Y \in T_M$ and $\xi, \eta \in T_M^*$. We will denote this metric, which has (d, d)-signature, by I. A subbundle $L \subset T_M \oplus T_M^*$ is *isotropic* if $(U, V) = 0$ for every $U, V \in L$. If it has the maximal dimension d, it is called *maximally isotropic*. A subbundle L is *integrable* if the *Courant bracket*,

$$[U, V] = [X + \xi, Y + \eta] = [X, Y]_L + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi) \,, \tag{5.1}$$

is closed on L, i.e. if it is involutive. Here, $[.,.]_L$ is the Lie bracket on T_M and $\mathcal{L}$ is the Lie-derivative. The Courant bracket can be twisted by a closed 3-form H as follows

$$[U, V]_H = [X + \xi, Y + \eta]_H = [X, Y]_L + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi) + i_X i_Y H \,. \tag{5.2}$$

A subbundle L that is involutive under the H-twisted Courant bracket is called H-integrable. The only symmetries of the Lie bracket are diffeomorphisms. The Courant bracket however has an extra symmetry which is called the b-transform[5]:

$$e^b(X + \xi) = X + \xi - i_X b = X + \xi + bX \,. \tag{5.3}$$

[5]Note that our sign convention for the b-transform differs from the one in the generalized complex structure literature. In this convention one can represent the b-transform as the matrix $\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ working on $(X\ \xi)^T$.

JHEP08(2005)099

Under the b-transform the Courant bracket changes as

$$[e^b U, e^b V]_H = e^b [U,V]_{H+db}\,. \tag{5.4}$$

Therefore the b-transform is an automorphism if and only if $db = 0$. Taking $H' = H + db$ in eq. (5.4) we see that if L is H'-integrable, then $e^b L$ will be $(H' - db)$-integrable.

An element $U \in T_M \oplus T_M^*$ has a natural action on a sum of forms of different dimensions (henceforth just called form), $\phi \in \wedge^\bullet T_M^*$ as follows:

$$U \cdot \phi = (X, \xi)\cdot \phi = i_X \phi + \xi \wedge \phi\,. \tag{5.5}$$

In fact, this makes $T_M \oplus T_M^*$ a realization of the Clifford algebra Cliff(d,d) and the forms the spin representation since $(X+\xi)^2 \cdot \phi = (i_X \xi)\phi = (X+\xi, X+\xi)\phi$. A spinor ϕ is called *pure* if its *null space* $L_\phi = \{U \in T_M \oplus T_M^* \,:\, U \cdot \phi = 0\}$ is maximally isotropic. Every maximally isotropic subbundle L is represented by a unique pure spinor line U_L (i.e. a spinor defined up to a proportionality factor). If ϕ is a pure spinor of L, then $e^b\phi$ will be a pure spinor of $e^b L$. Pure spinors have a definite *parity.* They are positive if they consist solely of even forms and negative if they consist of odd forms.

In [18] it was shown that L is H-integrable if and only if for any spinor ϕ of the corresponding pure spinor line there exists a $U = (X, \xi)$ such that it satisfies $d_H \phi = (d + H\wedge)\phi = i_X\phi + \xi \wedge \phi$. In many examples, it will be possible to find a pure spinor such that simply $d_H\phi = 0$. In that case we find indeed that $d_{H-db}(e^b L) = 0$ such that $e^b L$ is $(H - db)$-integrable.

Let us now interpret eq. (3.1) in terms of a maximally isotropic subbundle. The $(1,1)$-tensor r defines a distribution $E \subseteq T_M$ consisting of the vector fields v satisfying $rv = v$. If the distribution is involutive with respect to the Lie bracket, through a point x we can define a submanifold N such that $E|_N = T_N$. We consider now a D-brane $(N, \mathcal{F})$ and define F as in eq. (3.3). Following [16] we introduce

$$\psi^i = \psi_L^i + \psi_R^i\,, \qquad \rho_i = g_{ij}\left(\psi_L^j - \psi_R^j\right). \tag{5.6}$$

The gluing condition (3.1) can then be rewritten as

$$\mathcal{R}\Psi = \Psi, \tag{5.7}$$

with

$$\mathcal{R} = e^F r e^{-F} = \begin{pmatrix} \mathbf{1} & 0 \\ F & \mathbf{1} \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & -r^t \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ -F & \mathbf{1} \end{pmatrix}. \tag{5.8}$$

and $\Psi = (\psi\ \rho)^T$. In fact, this means that Ψ belongs to the generalized tangent bundle (N, F), which is defined as

$$L(N,F) = \{X + \xi \in T_N \oplus T_M^* \,:\, \xi|_N = i_X F\} = e^F\left(T_N \oplus \mathrm{Ann}\, T_N\right). \tag{5.9}$$

This is a maximally isotropic subbundle. In [17] it is shown that the generalized tangent bundle is involutive with respect to the H-twisted Courant bracket precisely if E is involutive and $dF = -H|_N$. The corresponding pure spinor is given by

$$\tau_{(N,F)} = c \exp F \det \mathrm{Ann}\, T_N\,, \tag{5.10}$$

JHEP08(2005)099

where $\det \mathrm{Ann}\, T_N = \theta_1 \wedge \ldots \wedge \theta_{d-p}$ with $(\theta_1, \ldots, \theta_{d-p})$ a basis for $\mathrm{Ann}\, T_N$. $c \neq 0$ is an arbitrary constant. We see that in fact $\rho(F)$ defined in eq. (2.17) and rewritten in the manner of eq. (3.8) reads

$$\rho(F) = \sqrt{P(g)}\,\mathrm{se}(-\not{F})\Gamma_N = \sqrt{P(g)}\,\mathrm{se}(-\not{F})\Gamma_N^{\perp}\gamma_{1\ldots d}\,, \tag{5.11}$$

where we defined $\Gamma_N^{\perp}\gamma_{1\ldots d} = \Gamma_N$. By going to a coordinate system where the tangent directions to the submanifold are denoted by the first p coordinates, one easily sees that $\Gamma_N^{\perp}$ will be a product of γ-matrices in the normal space. Eqs. (5.10) and (5.11) show that $\tau_{(N,F)}$, the pure spinor associated to the generalized tangent bundle defined by $\mathcal{R}$, and $\rho(F)$, the spinor representation of the R in (3.1) are closely related. To be precise:

$$\rho(F) = \sqrt{P(g)}(-1)^{\frac{(d-p)(d-p-1)}{2}} \sum_l \frac{1}{l!}\left({}^{\star}\tau_{(N,F)}\right)^{j_1\ldots j_l}\gamma_{j_1\ldots j_l}\,. \tag{5.12}$$

An *almost generalized complex structure* is a map $\mathcal{J} : T_M \oplus T_M^* \to T_M \oplus T_M^*$ such that $\mathcal{J}^2 = -\mathbf{1}$ and that $\mathcal{J}$ is compatible with the metric: $(\mathcal{J}U, \mathcal{J}V) = (U, V)$. Let L and $\bar{L}$ denote the $(+i)$ and $(-i)$-eigenbundles of $\mathcal{J}$ respectively. L and $\bar{L}$ are maximally isotropic subbundles. $\mathcal{J}$ is an H-twisted generalized complex structure if and only if L is H-integrable (which implies that $\bar{L}$ is also H-integrable). We denote the pure spinor associated to L by $\phi_{\mathcal{J}}$.

An *H-twisted generalized Kähler structure* is a pair $(\mathcal{J}_1, \mathcal{J}_2)$ of commuting H-twisted generalized complex structures such that $G = IF = -I\mathcal{J}_1\mathcal{J}_2$ is a positive definite metric on $T_M \oplus T_M^*$. Note that $F = -\mathcal{J}_1\mathcal{J}_2$ satisfies $F^2 = \mathbf{1}$. We will call the $(+1)$ and (-1)-eigenbundles of F, C_+ and C_- respectively. We can define the projections $p_\pm = \frac{1}{2}(1+F)$ on $C_\pm$ and the projection π on T_M such that $\pi(X,\xi) = X$. It is easy to see that $\mathcal{J}_1 = \mathcal{J}_2$ on C_+ and $\mathcal{J}_1 = -\mathcal{J}_2$ on C_-. By projection from $C_\pm$, $\mathcal{J}_1$ induces two almost complex structures on M, which we denote $J_{L/R}$. More concretely, they are defined such that

$$\begin{aligned} \mathcal{J}_1 &= \pi|_{C_+}^{-1} J_L \pi p_+ + \pi|_{C_-}^{-1} J_R \pi p_-\,, \\ \mathcal{J}_2 &= \pi|_{C_+}^{-1} J_L \pi p_+ - \pi|_{C_-}^{-1} J_R \pi p_-\,. \end{aligned} \tag{5.13}$$

We see that since mirror symmetry sends $J_R \to -J_R$ it interchanges $\mathcal{J}_1$ and $\mathcal{J}_2$. In coordinates the metric $G = IF$ has the form

$$F = e^b g e^{-b} = \begin{pmatrix} \mathbf{1} & 0 \\ b & \mathbf{1} \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ -b & \mathbf{1} \end{pmatrix}, \tag{5.14}$$

and is thus determined by the pair (g, b). Furthermore

$$\mathcal{J}_{1/2} = \frac{1}{2} e^b \begin{pmatrix} J_L \pm J_R & -(\omega_L^{-1} \mp \omega_R^{-1}) \\ \omega_L \mp \omega_R & -(J_L^T \pm J_R^T) \end{pmatrix} e^{-b}\,, \tag{5.15}$$

with as before $\omega_{L/R} = g J_{L/R}$. The two pure spinors $\mathcal{J}_1$ and $\mathcal{J}_2$ are of the same parity if $n = d/2$ even and of opposite parity if n odd.

JHEP08(2005)099

In [17] it is shown that the generalized Kähler geometry $(\mathcal{J}_1, \mathcal{J}_2)$ is completely equivalent to the bihermitian geometry (g, J_L, J_R, H) namely $\mathcal{J}_1$ and $\mathcal{J}_2$ defined in (5.15) are $\tilde{H}$-integrable with $\tilde{H} = H - db$ if and only if the corresponding J_L and J_R from (5.13) satisfy

$$(\nabla \pm \frac{1}{2}H)J_{L/R} = 0\,, \tag{5.16}$$

and H is of type $(2,1) + (1,2)$ with respect to both $J_{L/R}$ (the latter condition is implied by $J_{L/R}$ integrable). Note that to each bihermitian structure correspond many generalized Kähler structures which differ from each other by a b-transform. In what follows we apply a b-transform such that $b = 0$ in (5.14) and (5.15). This means that we will put all b-dependence into $\mathcal{R}$ defined in (5.8).

The canonical example is a usual Kähler structure (g, J, ω) with $\omega = gJ$. The generalized complex structures and metric are

$$\mathcal{J}_1 = \begin{pmatrix} J & 0 \\ 0 & -J^T \end{pmatrix}, \qquad \mathcal{J}_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}, \qquad G = -I\mathcal{J}_1\mathcal{J}_2 = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}. \tag{5.17}$$

$\mathcal{J}_1$ is Courant integrable if and only if J is Lie integrable and $\mathcal{J}_2$ is integrable if and only if $d\omega = 0$ so that we indeed find a Kähler structure. The corresponding pure spinors are $\bar{\Omega}$ and $e^{-i\omega}$. From eq. (5.15) we find $J_L = J_R$. Note that putting $J_L = -J_R$ corresponds to switching $\mathcal{J}_1 \leftrightarrow \mathcal{J}_2$. These are the two special cases under study in section 4.

With the choice $b = 0$, vectors of $C_\pm$ have the form $(v^a, \pm g_{ab}v^b)$ respectively. From eq. (5.6) we see that $\Psi = (\psi, \rho) = (\psi_L, g\psi_L) + (\psi_R, -g\psi_R)$ so that the part with ψ_L belongs to C_+ and the part with ψ_R to C_-. In [16] it is shown that

$$R^{-1}J_R R = \pm J_L \Leftrightarrow \mathcal{J}_{1/2} = \mathcal{R}^{-1}\mathcal{J}_{1/2}\mathcal{R}\,. \tag{5.18}$$

So using eq. (3.16) we see that the first condition for the saturation of the calibration bound (2.24a) means that the generalized tangent bundle $L(N, F)$ of the D-brane should be invariant under $\mathcal{J}_{1/2}$ or $\mathcal{J}_2$. This implies the generalized submanifold should be in fact a generalized complex submanifold. As studied in section 4 in the case of a Calabi-Yau manifold this leads to complex and coisotropic D-branes respectively.

We will study the second condition in the next subsection, but first we need to introduce the pure spinors of $\mathcal{J}_{1/2}$. As in [19, 20] it will be convenient to explore the connection between forms on M and Cliff(d) bispinors:

$$C = \sum_l \frac{1}{l!} C^{[l]}_{i_1 \ldots i_l} dx^{i_1} \wedge \ldots \wedge dx^{i_l} \longleftrightarrow \not{C} = \sum_l \frac{1}{l!} C^{[l]}_{i_1 \ldots i_l} \gamma^{i_1 \ldots i_l}\,. \tag{5.19}$$

Because of (5.5) we have a relation between Cliff(d, d) spinors and sums of forms which are thus in turn related to Cliff(d) bispinors. We can act on the bispinor with γ-matrices from the left (denoted by $\overrightarrow{\gamma_i}$) and γ-matrices from the right (denoted by $\overleftarrow{\gamma_i}$), which identifying C and $\not{C}$ has the effect:

JHEP08(2005)099

$$
\begin{aligned}
X^j \overrightarrow{\gamma_j} &\longleftrightarrow X^j i_j + g_{ij} X^j dx^i \wedge = (X, gX)\cdot\,, \\
X^j \overleftarrow{\gamma_j} &\longleftrightarrow (-1)^{P+1} \left(X^j i_j - g_{ij} X^j dx^i \wedge \right) = (-1)^{P+1} (X, -gX)\cdot\,,
\end{aligned}
\tag{5.20}
$$

with $(-1)^P$ the parity of the pure spinor. So we see that $X^j \overrightarrow{\gamma_j}$ reproduces the action of elements of C_+ and $X^j \overleftarrow{\gamma_j}$ of elements of C_-. It follows immediately that $\eta_L\, \eta_R^\dagger$ is annihilated by $(+i)$-eigenvalues of J_L on the C_+-side and $(-i)$-eigenvalues of J_R on the C_--side. We see from eq. (5.13) that it must corresponds to $\mathcal{J}_2$. Analogously $\eta_L\, (\eta_R^c)^\dagger$ corresponds to $\mathcal{J}_1$. Fierzing one finds

$$
\not{\phi}_{\mathcal{J}_2} = \eta_L\, \eta_R^\dagger = \frac{1}{\dim S} \sum_l \frac{(-1)^{\frac{l(l-1)}{2}}}{l!} \left(\eta_R^\dagger \gamma_{i_1 \ldots i_l} \eta_L \right) \gamma^{i_1 \ldots i_l}\,, \tag{5.21}
$$

with $\dim S$ the dimension of the spinor representation. We find the analogous expression for $\mathcal{J}_1$ by replacing $\eta_R \to \eta_R^c$. From eq. (2.11) it follows that

$$
d \left(e^{-\Phi} \phi_{\mathcal{J}_{1/2}} \right) + H \wedge e^{-\Phi} \phi_{\mathcal{J}_{1/2}} = 0\,, \tag{5.22}
$$

which means that both generalized complex structures are not only H-integrable but in fact H-twisted generalized Calabi-Yau structures according to Hitchin's definition [18]. Note that Hitchin's definition does not require a generalized Kähler structure, just one generalized complex structure.

However, we can also show that we have a generalized Calabi-Yau structure in the sense of Gualtieri [17], which is rather a generalized Kähler structure so that both pure spinors $e^{-\Phi} \phi_{\mathcal{J}_{1/2}}$ satisfy eq. (5.22) and their lengths are related by a constant $c \in \mathbb{R}$:

$$
(\phi_{\mathcal{J}_1}, \bar{\phi}_{\mathcal{J}_1}) = c (\phi_{\mathcal{J}_2}, \bar{\phi}_{\mathcal{J}_2})\,, \tag{5.23}
$$

where we defined the $\mathrm{Spin}_0(d,d)$-invariant bilinear form on spinors:

$$
(\phi_1, \phi_2) = (\alpha(\phi_1) \wedge \phi_2)_{\mathrm{top}}\,. \tag{5.24}
$$

Here α reverses the indices of a form:

$$
\alpha(\phi) = \sum_l \phi_{i_l \ldots i_1} dx^{i_1} \wedge \ldots \wedge dx^{i_l}\,. \tag{5.25}
$$

In particular the bilinear form is invariant under the b-transform. To show eq. (5.23) we note that for two form ϕ_1 and ϕ_2:

$$
\mathrm{Tr} \left(\not{\phi}_1 \gamma_{1 \ldots d} \not{\phi}_2 \right) = \frac{1}{\sqrt{g}} \left(\alpha(\phi_1) \wedge \alpha(\phi_2) \right)_{\mathrm{top}, \epsilon}\,. \tag{5.26}
$$

To prove this equation we used the fact that all antisymmetrized products of γ-matrices are traceless so that the trace selects out the piece proportional to $\mathbf{1}$. Now we plug in $\phi_{\mathcal{J}_1}$ from eq. (5.21) and use $\left(\eta_R^\dagger \gamma_{i_1 \ldots i_l} \eta_L \right)^* = \left(\eta_L^\dagger \gamma_{i_l \ldots i_1} \eta_R \right)$ to calculate $\bar{\phi}_{\mathcal{J}_1}$. We do the same for $\phi_{\mathcal{J}_2}$. In the end we find simply

$$
c = (-1)^n\,. \tag{5.27}
$$

JHEP08(2005)099

5.2 Generalized calibrations in generalized Calabi-Yau manifolds

With all this machinery we are finally ready to show that the notion of generalized calibrations introduced here is equivalent to definition 7.10 of [17]. We start from (2.24b) and introduce a trace over the spinor representation on the right hand side. This is trivial because the right hand side is a scalar. Next we use cyclicity:

$$\sqrt{\det(P(g)+F)} = e^{-i\gamma}\,\mathrm{Tr}\left(\eta_R^\dagger \rho(F)\eta_L\right) = e^{-i\gamma}\,\mathrm{Tr}\left(\eta_L\eta_R^\dagger\,\rho(F)\right). \tag{5.28}$$

Now we can plug in eqs. (5.12) and (5.21) and use eq. (5.26). We find:

$$\sqrt{\det(P(g)+F)} = (-1)^t e^{-i\gamma}\sqrt{P(g)}\frac{1}{\sqrt{g}}\left(\phi_{\mathcal{J}_2},\tau_{(N,F)}\right)_\epsilon\,, \tag{5.29}$$

where $(-1)^t = (-1)^{(d-p)(d-p-1)/2}(-1)^{p(d-p)}$ is a sign.

The volume form $\mathrm{vol}|_{L(N,F)}$ contains $\sqrt{\det G|_{L(N,F)}}$ where $G|_{L(N,F)}$ is the pull-back of the metric G to the generalized tangent bundle $L(N,F)$. We calculate

$$\sqrt{\det G|_{L(N,F)}} \propto \det(P(g)+F)\,. \tag{5.30}$$

Plugging in eq. (5.29) we find

$$\sqrt{\det G|_{L(N,F)}} \propto (\tau_{(N,F)},\phi_{\mathcal{J}_2})(\phi_{\mathcal{J}_2},\tau_{(N,F)})\,. \tag{5.31}$$

In [17] the Cliff(d,d) bispinor Ω_2 is defined as $\Omega_2 = (.,\phi_{\mathcal{J}_2})(\phi_{\mathcal{J}_2},.) \in \wedge^\bullet(T_M\oplus T_M^*)\otimes \det T_M^*$. Gualtieri argues furthermore that for a pure spinor ϕ the bispinor $(.,\phi)(\phi,.)$ is an element of $\det L\otimes\det T_M^*$ where L is the corresponding maximal isotropic subbundle. Introducing again a trace, but now over the (d,d)-spinors we use this fact for $\tau_{(N,F)}$ and find

$$\mathrm{vol}|_{L(N,F)} = e^{i\gamma'}\frac{1}{\sqrt{g}}\Omega_2|_{L(N,F)}\,. \tag{5.32}$$

We introduced the $1/\sqrt{g}$ factor to compensate for the extra $\det T_M^*$ factor in the transformation law of Ω_2. Eq. (5.32) is precisely the definition of the calibration introduced in [17].

In the end we find that a D-brane is generalized calibrated if it is a generalized complex submanifold with respect to $\mathcal{J}_1$ and obeys eq. (5.32) for $\mathcal{J}_2$.

6. Discussion

In this paper we have introduced generalized calibrations that provide a bound on the Dirac-Born-Infeld energy, rather than the volume. We considered a supersymmetric background with non-vanishing dilaton and 3-form H. We showed that

$$e^{-\Phi}\sqrt{\det(P(g)+F)} \geq \Re\left(e^{-i\gamma}e^{-\Phi}\eta_R^\dagger\rho(F)\eta_L\right), \tag{6.1}$$

with $e^{-i\gamma}$ a constant phase and η_L and η_R generators of left- and right-moving unbroken supersymmetry. We established the relation between calibrated submanifolds and supersymmetric cycles. We showed that one obtains the same results by demanding supersymmetry in the string world-sheet approach. Finally, we made the connection with the

JHEP08(2005)099

calibrations introduced in [17] in the context of generalized Calabi-Yau geometry. This latter geometry contains two commuting generalized complex structures $(\mathcal{J}_1, \mathcal{J}_2)$. We showed that D-branes are calibrated, in the sense of this paper, if and only if they are complex submanifolds with respect to $\mathcal{J}_1$ and calibrated, in the sense of [17], with respect to $\mathcal{J}_2$, or vice-versa. Furthermore, we note that the mirror map changes the sign of the U(1) R-symmetry in the right-moving sector and thus sends $J_R \to -J_R$ and changes η_R into η_R^c. It also interchanges $\mathcal{J}_1$ and $\mathcal{J}_2$. Thus B- and A-type branes are interchanged.

We would like to emphasize that our analysis only applies to abelian gauge fields. It would be interesting although presumably extremely hard to generalize to non-abelian gauge fields. In this case the full form of the analogue of the Dirac-Born-Infeld action is not even known and there is an intricate interplay with derivative corrections.

In this paper we considered $\mathrm{SU}(n)_L \times \mathrm{SU}(n)_R$ structure. We do not have to restrict ourselves to this case. In fact, the analysis in section 2 goes through as soon as we have an η_L on the left and a η_R on the right satisfying eqs. (2.9). We can then construct a closed calibration, establish the calibration bound, and show the relation with supersymmetry as before. So we could also consider for example $\mathrm{Spin}(7) \times \mathrm{Spin}(7)$ in $d = 8$ and $G_2 \times G_2$ in $d = 7$. In the latter case we could make the connection with the generalized G_2-structures introduced in [42, 30]. We could define a calibration in the same way as for a generalized Calabi-Yau geometry. The details remain work in progress.

We saw that for getting supersymmetric D-branes from the world-sheet perspective one also needed to impose stability as an additional condition in addition to just being topological. However, stability is also important even if one stays within topological string theory. Indeed, at the quantum level there is an anomaly in the R-charge. For A-branes without gauge field the anomaly vanishes if the Maslov class vanishes (for a review see section 3.1.1 of [43] or sections 38.4 and 39.3 of [44]). The definition of the Maslov class is closely related to the notion of "special" in special lagrangian. Specifically, it is easy to show that any special lagrangian submanifold has vanishing Maslov class. Although the conditions for anomaly cancellation for the coisotropic A-branes are not known, in [14] a proposal for a generalized Maslov class was made. Even more speculatively, we could also make here a proposal for a generalized Maslov class for the case $H \neq 0$. Topological branes satisfy eq. (2.24a) or rephrased conveniently:

$$\sqrt{\det(P(g) + F)} = e^{-i\beta(\sigma)} \eta_R^\dagger \rho(F) \eta_L \,, \tag{6.2}$$

where $e^{-\beta(\sigma)}$ provides a map from the submanifold N on which the D-brane wraps to the circle S^1. This in turn induces a map on the fundamental group $\beta_* : \pi_1(N) \to \pi_1(S^1) \cong \mathbb{Z}$ which we could take as the generalized Maslov class. Clearly, if the brane is stable, this Maslov class is trivial.

Another interesting generalization would be to reintroduce the R-R fields in the supersymmetry transformations (2.9). The exterior derivative of the calibration will now be related to these R-R fields (see [20, 21]). It would be nice to introduce a new calibration so that calibrated submanifolds will minimize the sum of the Dirac-Born-Infeld term and the Wess-Zumino term.

JHEP08(2005)099

As a final speculation, we note that in [45] a relation between calibrations and the effective superpotential was found in the context of compactification on Calabi-Yau 4-folds. It would be interesting to see if this relation could be extended to generalized calibrations and generalized Calabi-Yau manifolds and to compare with the results of [21].

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A. Conventions

We use $\mu, \nu, \ldots$ for space-time indices, $i, j, \ldots$ for indices on the internal manifold M and $a, b, \ldots$ for indices on the D-brane world-volume. d indicates the dimension of M and p the dimension of the D-brane. If d even, we define $n = d/2$ and if p even, $k = p/2$.

Given a form $\phi = \frac{1}{l!}\phi_{i_1 \ldots i_l} dx^{i_1} \ldots dx^{i_l}$ we will denote its contraction with the *epsilon*-tensor as

$$\phi_\epsilon = \frac{1}{l!}\phi_{i_1 \ldots i_l}\epsilon^{i_1 \ldots i_l} , \tag{A.1}$$

and the contraction with γ-matrices as

$$\not{\phi} = \frac{1}{l!}\phi_{i_1 \ldots i_l}\gamma^{i_1 \ldots i_l} . \tag{A.2}$$

The Hodge dual form is given by

$$^\star\phi^{j_1 \ldots j_{d-l}} = \frac{1}{\sqrt{g}\, l!}\phi_{i_1 \ldots i_l}\epsilon^{i_1 \ldots i_l j_1 \ldots j_{d-l}} . \tag{A.3}$$

γ-matrices on curved space are as usual defined as $\gamma_i = e_i^A \gamma_A$, where e_i^A is the vielbein and γ_A a γ-matrix on flat space with metric δ_{AB}. γ-matrices γ_a on the D-brane world-volume are defined as the pull-back of γ-matrices on the internal manifold: $\gamma_a = \partial_a x^i \gamma_i$. Furthermore we define

$$\gamma_{1\ldots d} = \frac{1}{\sqrt{g}}\gamma_{i_1 \ldots i_d}\epsilon^{i_1 \ldots i_d} . \tag{A.4}$$

Since $\gamma_{1\ldots d}^2 = (-1)^{\frac{d(d-1)}{2}}$ we should define the chirality matrix as

$$\gamma_{d+1} = i^{\frac{d(d-1)}{2}}\gamma_{1\ldots d} . \tag{A.5}$$

Given a complex structure J, the spinor η_0 — the *empty state* — is such that it is annihilated by all $X^i\gamma_i$ where the vector X is a $(+i)$-eigenvalue of J. If we choose complex coordinates $\alpha, \beta, \ldots$ such that

$$J^\alpha{}_\beta = i\delta^\alpha{}_\beta \qquad J^{\bar{\alpha}}{}_{\bar{\beta}} = -i\delta^{\bar{\alpha}}{}_{\bar{\beta}}, \tag{A.6}$$

JHEP08(2005)099

η_0 is annihilated by all γ_α:

$$\gamma_\alpha \eta_0 = 0, \quad \text{for all}\alpha \tag{A.7}$$

The *completely filled state* η_0^c on the other hand is given by

$$\eta_0^c = \frac{1}{g^{1/4} 2^{n/2}} \gamma_{\bar{n}\ldots\bar{1}} \eta_0 \,. \tag{A.8}$$

It satisfies $\gamma_{\bar{\alpha}} \eta_0^c = 0$ for all α.

Starting from the complex coordinates we can also define coordinates x^i, y^i in which J takes a block-diagonal form:

$$z^\alpha = \frac{1}{\sqrt{2}} (x^\alpha + i y^\alpha) \,, \qquad z^{\bar{\alpha}} = \frac{1}{\sqrt{2}} (x^\alpha - i y^\alpha) \,, \tag{A.9}$$

Building the spinor representation by acting with $\gamma_{\bar{\alpha}}$ on η_0, we find that the γ_α and $\gamma_{\bar{\alpha}}$ are real in this representation such that $\gamma^*_{x^\alpha} = \gamma^*_{x^\alpha}$ and $\gamma^*_{y^\alpha} = -\gamma^*_{y^\alpha}$. We can then define the charge conjugation matrix C as

$$\begin{aligned} C &= \frac{1}{g^{1/4}} \gamma_{x^n \ldots x^1} \,, \qquad \text{for}\, n\, \text{odd} \,, \\ C &= \frac{1}{g^{1/4}} (-1)^{n/2} \gamma_{y^n \ldots y^1} \,, \qquad \text{for}\, n\, \text{even} \,, \end{aligned} \tag{A.10}$$

which indeed obeys

$$(\gamma_i)^* = C^{-1} \gamma_i C \,. \tag{A.11}$$

From (A.8) and (A.10) follows

$$\eta^c = C \eta^* \,. \tag{A.12}$$

From the defining property (A.11) follows that $CC^* = \delta \mathbf{1}$ with $\delta = \delta^*$. With our choices C is normalized such that $|\delta| = 1$. To be precise

$$CC^* = (-1)^{\frac{n(n-1)}{2}} \mathbf{1} \,. \tag{A.13}$$

Note that although we used a specific choice of γ-matrices and C the remaining sign is independent of that.

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JHEP08(2005)099